MODEL ANSWERS

Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number	Candidate Number

Thursday 14 May 2020

Afternoon

Paper Reference **8FM0/22**

Further Mathematics

Advanced Subsidiary
Further Mathematics options
22: Further Pure Mathematics 2
(Part of option A only)

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- The total mark for this part of the examination is 40. There are 5 questions.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

1/1/1/1/1/1/1/1/

Pearson

1. The set $G = \{1, 3, 7, 9, 11, 13, 17, 19\}$ under the binary operation of multiplication modulo 20 forms a group.

(a) Find the inverse of each element of G.

(3)

(b) Find the order of each element of G.

(3)

(c) Find a subgroup of G of order 4

(1)

(d) Explain how the subgroup you found in part (c) satisfies Lagrange's theorem.

(1)

a) Cayley table:

$\times_{20}$	1	3	7	9	11	13	17	19
1	1	3	7	9	11	13	17	19
3	3	9	1	7	13	19	11	17
7	7	1	9	3	17	11	19	13
9	9	7	3	1	9	17	13	11
11	11	13	17	19	1	3	7	9
13	13	19	11	17	3	9	1	7
17	17	11	19	13	7	1	9	3
19	19	17	13	11	9	7	3	1

inverse given when $a \times_n b = 1$

1, 9, 11 & 19 are self-inverse.

element	inverse
3	7
7	3
13	17
17	13

b) order is smallest +ve integer k such that $a^k = 1 \bmod 20$

$1^1 = 1 \bmod 20 \Rightarrow$ order 1

$3^4 = 81 = 1 \bmod 20 \Rightarrow$ order 4

$7^4 = 2401 =$ " " $\Rightarrow$ order 4

$9^2 = 81 =$ " " $\Rightarrow$ order 2

$11^2 = 121 =$ " " $\Rightarrow$ order 2

$13^4 = 28561 =$ " " $\Rightarrow$ order 4

$17^4 = 83521 =$ " " $\Rightarrow$ order 4

$19^2 = 361 =$ " " $\Rightarrow$ order 2

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Question 1 continued

c) order 4 → 4 elements from G

subgroup must have closure, identity, inverses, & associativity

{1, 3, 7, 9}   self-inverse

identity   3 & 7 are inverses of each other

X_{20} is an associative operator

d) to satisfy Lagrange's theorem, |H| divides |G|

4 is a factor of 8 ∴ L.T. satisfied

Question 1 continued

Question 1 continued

(Total for Question 1 is 8 marks)

2. The highest common factor of 963 and 657 is c.

(a) Use the Euclidean algorithm to find the value of c.

(3)

(b) Hence find integers a and b such that

$$963a + 657b = c$$

(3)

a) $a = bq_1 + r_1$: $963 = 657 \times 1 + 306$

$b = q_2 r_1 + r_2$: $657 = 306 \times 2 + 45$

$306 = 45 \times 6 + 9$

$45 = 36 \times 1 + 9$

$36 = 9 \times 4 + 0$ ← no remainder ⇒ algorithm complete

$\therefore$ HCF$(963, 657) = 9 = c$

b) reverse above process: $9 = 45 - 36 \times 1$

$36 = 306 - 45 \times 6 \xrightarrow{\text{sub in}} 9 = 45 - (306 - 45 \times 6)$

$9 = 7 \times 45 - 306$

$45 = 657 - 2 \times 306 \Rightarrow 9 = 7 \times (657 - 2 \times 306) - 306$

$= 7 \times 657 - 15 \times 306$

$306 = 963 - 1 \times 657 \Rightarrow 9 = 7 \times 657 - 15 \times (963 - 1 \times 657)$

$\therefore \quad 9 = -15 \times 963 + 22 \times 657$

$a = -15$, $b = 22$

Sub in repeatedly until you reach 963 & 657

Question 2 continued

Turn over ▶

Question 2 continued

Question 2 continued

(Total for Question 2 is 6 marks)

3. (i)

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$$

(a) Show that the characteristic equation for **A** is $\lambda^2 - 5\lambda + 6 = 0$

(2)

(b) Use the Cayley-Hamilton theorem to find integers p and q such that

$$\mathbf{A}^3 = p\mathbf{A} + q\mathbf{I}$$

(3)

(ii) Given that the 2×2 matrix **M** has eigenvalues $-1 + \mathrm{i}$ and $-1 - \mathrm{i}$, with eigenvectors $\begin{pmatrix} 1 \\ 2-\mathrm{i} \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2+\mathrm{i} \end{pmatrix}$ respectively, find the matrix **M**.

(5)

i. a) C.E. given by $\det(A - \lambda \mathbb{1}) = 0$: $\begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)+2 \overset{!}{=} 0$

$\Rightarrow 4 - 5\lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 5\lambda + 6 = 0$

b) C-H theorem : every square matrix satisfies its own C.E.

$\hookrightarrow A^2 - 5A + 6\mathbb{1} = 0 \Rightarrow A^2 = 5A - 6\mathbb{1}$

$\times A$: $A^3 - 5A^2 + 6A = 0$

$\hookrightarrow A^3 = 5(5A - 6\mathbb{1}) - 6A$

$= 25A - 30\mathbb{1} - 6A$

$\Rightarrow A^3 = 19A - 30\mathbb{1}$

Question 3 continued

ii. eigenvalues given by $\det(M - \lambda I) = 0$

eigenvectors given by $M\underline{v} = \lambda\underline{v}$

let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2-i \end{pmatrix} = (-1+i)\begin{pmatrix} 1 \\ 2-i \end{pmatrix}$

$\hookrightarrow a + b(2-i) = -1+i$ ①

$c + d(2-i) = (-1+i)(2-i) = -2+i+2i+1 = -1+3i$ ②

also $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2+i \end{pmatrix} = (-1-i)\begin{pmatrix} 1 \\ 2+i \end{pmatrix}$

$\hookrightarrow a + b(2+i) = -1-i$ ③

$c + d(2+i) = -1-3i$ ④

① - ③: $-2bi = +2i \Rightarrow b = -1$

real parts of ①: $a + 2(-1) = -1 \Rightarrow a = 1$

② - ④: $-2di = 6i \Rightarrow d = -3$

real parts of ②: $c + 2(-3) = -1 \Rightarrow c = 5$

$\therefore$ $\boxed{M = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}}$

Question 3 continued

Question 3 continued

(Total for Question 3 is 10 marks)

4. Sam borrows £10 000 from a bank to pay for an extension to his house.
The bank charges 5% annual interest on the portion of the loan yet to be repaid.
Immediately after the interest has been added at the end of each year and before the start of the next year, Sam pays the bank a fixed amount, £F.

Given that £A_n (where $A_n \geqslant 0$) is the amount owed at the start of year n,

(a) write down an expression for A_{n+1} in terms of A_n and F,

(1)

(b) prove, by induction that, for $n \geqslant 1$

$$A_n = (10\,000 - 20F)1.05^{n-1} + 20F$$

(5)

(c) Find the smallest value of F for which Sam can repay all of the loan by the start of year 16.

(4)

a) 5% interest on what is owed @ start of year n,

then subtract fixed value paid

↳ $\boxed{A_{n+1} = 1.05A_n - F}$

b) start with case n=1: $n=1 \Rightarrow A_1 = (10\,000 - 20F)1.05^{1-1} + 20F$

$\Rightarrow A_1 = 10\,000 - 20F + 20F = 10\,000$

10 000 is the amount paid in @ the start ∴ true for n=1

n=k: assume true for n=k, so

$$A_k = (10\,000 - 20F)1.05^{k-1} + 20F$$

write n=k+1 in terms of known expression, n=k:

$$A_{k+1} = 1.05(\underbrace{(10\,000 - 20F)1.05^{k-1} + 20F}_{A_k}) - F$$

↳ using $A_{n+1} = 1.05A_n - F$

Question 4 continued

$1.05 \times 20 = 21$

$\Rightarrow A_{k+1} = (10\,000 - 20F)1.05^k + 21F - F$

$= (10\,000 - 20F)1.05^{(k+1)-1} + 20F$

↳ in correct form, so result holds for $n = k+1$

$\therefore A_n = (10\,000 - 20F)1.05^{n-1} + 20F$ is true for all $n \geq 1$

c) repay all by 16^{th} year: $(10\,000 - 20F)1.05^{16-1} + 20F \leq 0$

$\Rightarrow 10\,000 \times 1.05^{15} \leq 20F(1.05^{15} - 1)$

$\Rightarrow F \geq \dfrac{10\,000 \times 1.05^{15}}{20(1.05^{15} - 1)}$

$\therefore$ smallest value of F is £963.43

Question 4 continued

Question 4 continued

(Total for Question 4 is 10 marks)

5.

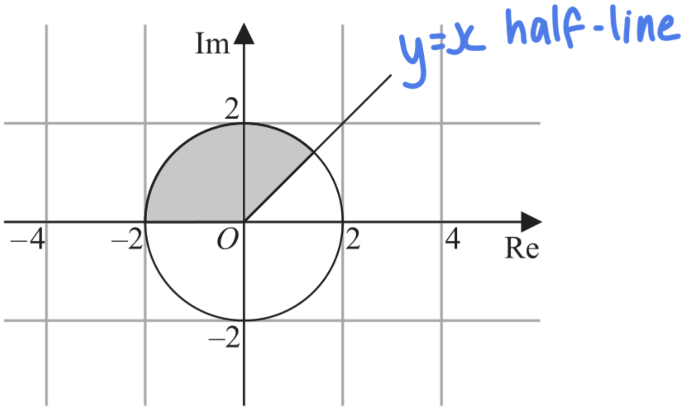


Figure 1

Figure 1 shows an Argand diagram.
The set of points, A, that lies within the shaded region, including its boundaries, is defined by

$$A = \{z : p \leqslant \arg(z) \leqslant q\} \cap \{z : |z| \leqslant r\}$$

where p, q and r are positive constants.

(a) Write down the values of p, q and r.

(2)

Given that $w = -2\sqrt{3} + 2\text{i}$ and $z \in A$,

(b) find the maximum value of $|w - z|^2$ giving your answer in an exact simplified form.

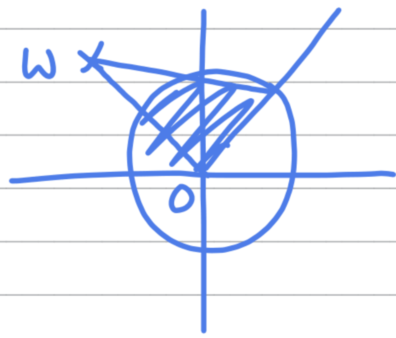
(4)

a) $y=x$ halfline $\Rightarrow$ $p = \frac{\pi}{4}$

other boundary is x-axis $\Rightarrow$ $q = \pi$

circle intersects axes @ 2 $\therefore$ $r = 2$, $|z| \leq 2$

b) w is outside z-region



max. distance when z is @ intersection

of $y=x$ & $x^2+y^2=4$

$\arg w = \tan^{-1}\left(-\frac{2}{2\sqrt{3}}\right) = -\frac{\pi}{6} + \pi = \frac{5}{6}\pi$

$\frac{5}{6}\pi - \frac{\pi}{2} = \frac{\pi}{3}$ $\leftarrow$ angle of w from Im. axis

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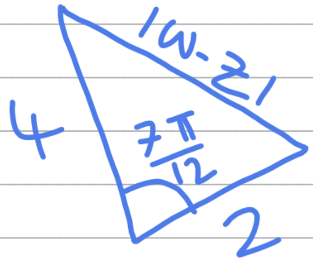
Question 5 continued

$\therefore$ angle between $y=x$ & $OW = \frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}$

$|w| = |(2\sqrt{3})^2 + 2^2| = 4$



cosine rule: $a^2 = b^2 + c^2 - 2bc \cdot \cos A$

$\Rightarrow d^2 = 4^2 + 2^2 - 2 \times 4 \times 2 \cos\left(\frac{7\pi}{12}\right)$

$= 20 - 4\sqrt{2} + 4\sqrt{6}$

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Question 5 continued

(Total for Question 5 is 6 marks)

TOTAL FOR FURTHER PURE MATHEMATICS 2 IS 40 MARKS